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DEVELOPMENT OF THE FUNDAMENTAL IDEAS OF THE DIFFERENTIAL CALCULUS.

BY G. H. GRAVES.

The purpose of this paper is to present the characteristic features of the different methods, by which the differential calculus has been introduced to students beginning the study of the subject.

The *first* traces of the science are to be found among the Greeks. In passing from commensurable to incommensurable figures, from rectilinear to curvilinear, the Greek mathematicians were obliged to consider a process repeated indefinitely which gave rise to quantities indefinitely small. The ancient geometers surmounted this difficulty by introducing the "Method of Exhaustions," which may be illustrated by the following theorem from Euclid:

"Circles are to one another as the squares on their diameters."*

Let C and C' be two circles and D and D', their diameters. Also let L and G be any two areas such that $L < C'$ and $G > C'$. Then

$$C:L \neq D^2:D'^2,$$

because we may inscribe in C' a regular polygon $P' > L$, and if P is the similar polygon inscribed in C,

$$P:P' = D^2:D'^2,$$

where $$P < C \text{ and } P' > L.$$

Also

$$C:G \neq D^2:D'^2,$$

because we may circumscribe about C' a regular polygon $R' < G$, and if R is the similar polygon circumscribed about C,

$$R:R' = D^2:D'^2,$$

where $$R > C \text{ and } R' < G.$$

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* Heath's translation (Cambridge, 1908) Book xii., Prop. ii. The proof above is merely a summary of Euclid's.

Hence

$$C:C'=D^2:D'^2.$$

This is essentially what we still use in elementary geometry and is mathematically rigorous.

The *second* method of dealing with such problems is known in its present form as the "Method of Infinitesimals."* The beginnings of this method are seen in the work of Kepler† and Cavalieri.‡ These men considered a curve as a polygon with an infinite number of sides and based their deductions on this paradoxical substitute for the precise idea that a curve is the limit of the inscribed polygons as the number of their sides is increased. The culmination of this movement was seen in the work of Leibnitz.

He is chiefly interested in a "differential," dx, which he obtains on the metaphysical assumption that successive division results at last in something which can be no further subdivided. His dx is one of these indivisible parts of the x-axis, not zero, yet so small that two finite quantities differing by dx are to be regarded as interchangeable. In his letter to Wallis, March 30, 1699, he says:§

"It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x+dx$, dx is rejected. But it is different if we seek the difference between $x+dx$ and x, for then the finite quantities disappear. Similarly we cannot have xdx and $dxdx$ standing together. Hence if we are to differentiate xy we write:

$$(x+dx)(y+dy)-xy=xdy+ydx+dxdy.$$

But here $dxdy$ is to be rejected as incomparably less than $xdy+ydx$. Thus in any particular case the error is less than any finite quantity."

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* Examples of text-books using this method: Lodge, McMahon and Snyder.

† "Rechnung der Cörperlichen Figuren" (1616).

‡ "Geometria Indivisibilibus Continuorum" (1635).

§ "Leibnitzens Mathematische Schriften," Gerhardt ed., Vol. 4, p. 63 (Series III. in Leibnitzens Gesammelte Werke, Pertz ed., Halle, 1859).

This neglecting of quantities which, however small they be, are expressly declared not to be zero, is of course inadmissible in mathematics except as a method leading to results confessedly approximate. The philosopher Berkeley, while he did not challenge the accuracy of the results obtained by this method, called its loose reasoning severely to account.[*] He showed in a few cases that the correctness of results was due to compensating errors, and suggested that this is always the case. Others have carried out this suggestion in greater detail.[†]

The *third* method, that of "Fluxions," was discovered by Newton at about the same time as the one just mentioned. With a little explanation as to what is meant by velocity at a point, it has become our "Method of Rates."[‡] Newton considered a quantity in a state of change, *e. g.*, a curve, as it is being described by a moving point. He called the infinitesimal path traced in the infinitesimal time the "moment" of the "flowing quantity" (*i. e.*, of the variable), and the ratio of the moment to its corresponding time, the "fluxion" of the variable (*i. e.*, the velocity of the motion). He denoted the fluxion of x by $\dot{x}$. Hence he can state:[§] "The moments of flowing quantities are as the velocities of their flowing or increasing," which we should express:

$$\frac{dy}{dx} = \frac{dy}{dt} : \frac{dx}{dt}.$$

His use of moments and fluxions is seen in the following explanation from the Method of Fluxions (pp. 24, 25):

"If the moment of x be represented by the product of its celerity $\dot{x}$ into an indefinitely small quantity o (that is $\dot{x}o$), the moment of y will be $\dot{y}o$, since $\dot{x}o$ and $\dot{y}o$ are to each other as $\dot{x}$ and $\dot{y}$. Now since the moments as $\dot{x}o$ and $\dot{y}o$ are the indefinitely little accessions of the flowing quantities, x and y, by which these quantities are increased through the several indefinitely little intervals of time, it follows that those quantities, x and y, after any indefinitely small interval of time, become

[*] *The Analyst*, London, 1734.

[†] *E. g.*, Carnot, "Reflexions sur la metaphysique du Calcul infinitesimal" (Paris, 1801).

[‡] Text-books: Hayes (1704), Rice and Johnson, Hardy.

[§] "Method of Fluxims," translated by Colson (London, 1736), p. 24.

$x+\dot{x}o$ and $y+\dot{y}o$. And therefore the equation which at all times indifferently expresses the relation of the flowing quantities will as well express the relation between $x+\dot{x}o$ and $y+\dot{y}o$ as between x and y; so that $x+xo$ and $y+\dot{y}o$ may be substituted in the same equation for those quantities instead of x and y.

" Therefore let any equation

$$x^3 - ax^2 + axy - y^3 = 0$$

be given, and substitute $x+\dot{x}o$ for x and $y+\dot{y}o$ for y, and there will arise

$$\begin{aligned} & x^3 + 3x^2\dot{x}o + 3x\dot{x}\dot{x}oo + \dot{x}^3o^3 \\ & - ax^2 - 2ax\dot{x}o - a\dot{x}^2oo \\ & + axy + ax\dot{y}o + a\dot{x}oy + a\dot{x}o\dot{y}o \\ & - y^3 - 3\dot{y}oy^2 - 3\dot{y}^2ooy - \dot{y}^3o^3 = 0. \end{aligned}$$

Now by supposition,

$$x^3 - ax^2 + axy - y^3 = 0,$$

which therefore being expunged and the remaining terms being divided by o, there will remain

$$\begin{aligned} 3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + a\dot{y}x & \\ + a\dot{x}\dot{y}o - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0. & \end{aligned}$$

But whereas o is supposed to be infinitely little that it may represent the moments of quantities, the terms which are multiplied by it will be nothing in respect to the rest. Therefore I reject them and there remains:

$$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2.\text{”}$$

This is open to the same criticism as Leibnitz's work. Newton himself however improved upon it in his method of "first and last ratios."

This *fourth* method is now known as the " Method of Limits." Newton's use of it will appear in the following example from the introduction to his " Quadrature of Curves."*

" Let a quantity x flow uniformly and let it be required to find the fluxion of x^n. In the time in which x by flowing becomes

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* Tractatus de Quadratura Curvarum (London, 1704), Introduction.

$x + o$, the quantity x^n becomes $\overline{x+o}|^n$; *i. e.,* by the method of infinite series,

$$x^n + nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +, \text{ etc.},$$

and the increment o and

$$nox^{n-1} + \frac{nn-n}{2}oox^{n-2} +, \text{ etc.},$$

are to each other as 1 and

$$nx^{n-1} + \frac{nn-n}{2}ox^{n-2} +, \text{ etc.}$$

Now let the increment vanish and their last ratio will be 1 to nx^{n-1}."

In the same introduction he gives the familiar geometric interpretation of these ratios as the slopes of a secant through two points of a curve and of the tangent which it approaches as the two points approach coincidence. He remarks:

"If the points are distant from each other by an interval however small, the secant will be distant from the tangent by a small interval. That it may coincide with the tangent and the last ratio be found, the two points must unite and coincide altogether. In mathematics errors, however small, must not be neglected."

Newton was criticized* because he here keeps expressions got by contradicting the assumption on which the expressions were first obtained. Also as Lagrange observes:† "This method has the great inconvenience of considering quantities in the state where they cease, so to say, to be quantities." And in general the employment of vanishing quantities in this manner confuses the limit of a variable ratio in which both terms approach zero simultaneously, with the ratio of two zeros. It is clear that Newton himself understood the difference, however, for he says, in his "Principia":‡ "Ultimate ratios in which quantities vanish, are not, strictly speaking, ratios of ultimate quantities, but limits to which the ratio of these quantities decreasing without limit, approach, and which, though they can come nearer than any given difference whatever, they can neither pass over nor attain before the quantities have diminished indefinitely."

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* *E. g.,* by Berkeley in "The Analyst."

† "Théorie des Fonctions" (Paris, 1813), Introduction.

‡ "Philosophiæ Naturalis Principia Mathematica," Section I.

With this idea of limits, the differential calculus finds a starting point from which it may be rigorously developed, and this is the method generally followed in modern text-books.*

Lagrange objected to this method on the ground that it requires limits to be regarded "in a peculiar manner, if not contrary, at least foreign to the spirit of modern analysis."† He proposed a *fifth* plan of introducing the calculus by the purely algebraic means of "derived functions."

In referring to the work of his predecessors he says:‡ "When we consider these different methods, we find that their only aim is to give the means of obtaining separately, the first terms of the development of a function, and in detaching and isolating these, so to say, from the rest of the series." His plan may be briefly set forth in his own words:§

"When to the variable of a function we give any increment, we may, by the ordinary rules of algebra expand the function, if it is algebraic, according to the powers of the increment. The first term of the expansion will be the given or 'primitive' function; the following terms will be formed of different functions of the same variable multiplied by successive powers of the increment. These new functions depend only on the primitive function from which they are derived and may be called 'derived functions.' In general whatever be the primitive function, algebraic or not, it may always be expanded or considered to be expanded in the same manner and thus give rise to derived functions."

In many ways this is a simplification, but to be rigorous as a general method it requires the consideration of the problem whether all functions are developable into power series, and whether the series obtained are convergent. This requires limits again "regarded in a peculiar manner." But in those functions which expand into a finite series or an infinite series which is easily tested, this method is useful and is often used in text-books on algebra.||

Finally we must mention Cauchy who has placed the Method

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* *E. g.*, Todhunter (1864), Byerly, Osgood.

† Introduction to the Leçons sur le Calcul des Fonctions (1808).

‡ Ibidem.

§ Théorie des Fonctions, Introduction. (Paris, 1813.)

|| *E. g.*, Fine, Hall and Knight.

of Infinitesimals on a sound basis and relieved modern analysis of the necessity of being cumbered with $\frac{dy}{dx}$ as an inseparable ratio.

He states in the preface to his "Leçons sur le Calcul Infinitesimal": "My principal aim has been to reconcile rigor with the simplicity that results from the direct consideration of infinitely small quantities." This he does by defining an infinitesimal as a variable which has zero for its limit. He makes the transition from

$$\frac{dy}{dx} = \frac{f(x+i) - f(x)}{i} = f'(x)$$

to

$$dy = f'(x)dx$$

as follows:*

"Let $y = f(x)$ be a function of the independent variable x; i, an infinitesimal, and h a finite quantity. If we put $i = \alpha h$, α will be an infinitesimal and we shall have the identity

$$\frac{f(x+i) - f(x)}{i} = \frac{f(x+\alpha h) - f(x)}{\alpha h},$$

whence we derive

$$(1) \qquad \frac{f(x+\alpha h) - f(x)}{\alpha} = \frac{f(x+i) - f(x)}{i} h.$$

The limit toward which the first member of this equation converges when the variable α approaches zero, h remaining constant, is what we call the "differential" of the function $y = f(x)$. We indicate this differential by the characteristic, d, as follows:

$$dy \text{ or } df(x).$$

It is easy to obtain its value when we know that of the derived function, y' or $f'(x)$. In fact taking the limits of both members of equation (1), we have in general :

$$(2) \qquad df(x) = hf'(x).$$

In the particular case where $f(x) = x$, equation (2) reduces to

$$dx = h.$$

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* "Résumé des Leçons sur le Calcul Infinitesimal," Quatrième Leçon (Paris, 1823). [Œuvres Complètes, Ser. II., Tome IV., Paris, 1899.]

Thus the differential of the independent variable, x, is simply the finite constant, h. Substituting, equation (2) will become

$$df(x) = f'(x)dx,$$

or what amounts to the same thing

$$dy = y'dx."$$

The plan generally followed in text-books to-day is not to introduce infinitesimals until a clear concept of the derivative has been obtained from the Method of Limits. Owing to the greater ease of operating with infinitesimals, and their use in higher analysis, the beginner needs to gain familiarity with them, but at the outset even simplicity of operations must give place to clearness of concept and here the Method of Limits seems to offer the least difficulties. It will be observed that all the methods we have mentioned depend ultimately on limits. When the student has learned to associate with the derivative, $f'(x)$ or $D_x u$, the concrete idea of the slope of a tangent to a curve it is not hard to conceive of dx as a finite increment to the abscissa of a point, and dy as the segment cut off by the tangent on the increment to the ordinate. "The principal aim," says Professor Klein,* "is to convince the student that there is here nothing mysterious, but only simple things which every one can understand."

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* "Elementarmathematik vom Höheren Standpunkt aus" (Leipzig, 1908), p. 488.